L_∞ – LIMIT THEOREMS FOR MARKOV PROCESSES

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ABSTRACT

A necessary and sufficient condition for convergence of Markov processes L_∞ is given. As a consequence we get a theorem concerning the convergence of Harris processes.

1. Definitions and notations. A Markov process is defined to be a quadruple (X,Σ,m,P) where (X,Σ,m) is a finite measure space with a positive measure m and where P is an operator on $L_1(m)$ satisfying (i) P is a contraction: $\| P \| \leqq 1$. (ii) P is positive: if $0 \leqq u \in L_1(m)$ then $uP \geqq 0$. The operator adjoint to P is defined on $L_\infty(m)$. It will also be denoted by P but will be written to the left of its variable. Thus $\langle uP,f\rangle = \langle u,Pf\rangle$ for $u \in L_1(m)$, $f \in L_\infty(m)$.

A finitely additive set function will be called a *charge*.

The operator P acts on the space of the charges weaker than m (the adjoint space of $L_\infty(m)$) in the following form:

1.1) $$\nu P(f) = \nu(Pf), \qquad f \in L_\infty(m)$$

The operator P is called *ergodic* if:

(1.2) $$P1_A = 1_A \Rightarrow m(A) = 0 \quad\text{or}\quad m(A^c) = 0$$

P is said to be *conservative* if:

(1.3) $$m(A) > 0 \Rightarrow \sum_{n=1}^{\infty} P^n 1_A(x) = \infty \qquad \text{a.e.}$$

The charge ν is said to be *invariant* under P if:

(1.4) $$\nu P = \nu$$

In particular if ν is a measure it is called an *invariant measure*. Let ν be invariant

(1.5) $$\nu = \nu^+ - \nu^- \text{ then } \nu^+P = \nu^+ \text{ and } \nu^-P = \nu^-$$

* This paper is a part of the author's Ph.D. thesis to be submitted to the Hebrew University of Jerusalem. The author wishes to express his thanks to Professor S. R. Foguel for much valuable advice and encouragement.

Received March 2, 1969.

(see [2]) because $\nu P = \nu^+ P - \nu^- P = (\nu P)^+ - (\nu P)^- = \nu^+ - \nu^-$ thus $\nu^+ P \geqq \nu^+$ but $\nu^+ P(1) = \nu^+(P1) \leqq \nu^+(1)$, and this implies $\nu^+ P = \nu^+$, and $\bar{\nu}\ P = \bar{\nu}$.

Let $P^n = Q_n + R_n$ where Q_n is an integral operator with the kernel $q_n(x,y)$, and if K is any integral operator such that $0 \leqq K \leqq R_n$ then $K = 0$. The process (X, Σ, m, P) is said to be a *Harris process* if it is ergodic and conservative and $Q_n > 0$ for some integer n (see [1] Chapter V).

2. L_∞-limit theorems

THEOREM 1. *Let $f \in L_\infty(m)$, f is orthogonal to every invariant charge, i.e. $\nu P = \nu \Rightarrow \nu(f) = 0$, if and only if:*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k f \right\|_\infty = 0 \tag{2.1}$$

Proof. This condition is clearly necessary. Let us prove the sufficiency. Consider the closure of the range of the operator $I - P$, $\overline{(I-P)L_\infty(m)}$, its orthogonal complement is the set of the invariant charges. If f is orthogonal to the invariant charges, then by the Hahn-Bannach Theorem, $f \in \overline{(I-P)L_\infty(m)}$, so that there exists a function g with $\|f - g + Pg\|_\infty < \varepsilon$. Therefore:

$$\left\| \frac{1}{n} \sum_{k=1}^{n} P^k f \right\|_\infty \leqq \left\| \frac{1}{n} \sum_{k=1}^{n} P^k (f - g + Pg) \right\|_\infty + \left\| \frac{1}{n} \sum_{k=1}^{n} P^k (g - Pg) \right\|_\infty \leqq \varepsilon + \frac{2\|g\|_\infty}{n}$$

but $\dfrac{2\|g\|_\infty}{n}$ tends to zero and ε is arbitrary, hence

$$\left\| \frac{1}{n} \sum_{k=1}^{n} P^k f \right\|_\infty \xrightarrow[n\to\infty]{} 0.$$

REMARK. In [1] Chapter IV, it is proved that if $f \in L_\infty(m)$ and there is a sequence of integers $\{n_i\}$ such that

$$\sum_{i=1}^{\infty} P^{n_i} f \in L_\infty(m) \text{ then } \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k f \right\|_\infty = 0.$$

It is clear that this proposition follows from Theorem 1.

THEOREM 2. *Let (X, Σ, m, P) be a Harris process, let μ be an invariant measure, then for each $\varepsilon > 0$ there exists a set A with $m(A^c) < \varepsilon$ so that for every function $f \in L_\infty(m)$ which is orthogonal to μ and* supp $f \subset A$, *(for example: $f = 1_B - \mu(B)/\mu(A) \cdot 1_A$, $B \subset A$) we have:*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k f \right\|_\infty = 0 \tag{2.2}$$

Proof. There is an integer k so that $Q_k > 0$, hence P^k can be written a sum $P^k = \tilde{Q} + \tilde{R}$ where $\tilde{Q}$ is a positive integral operator with the bounded kernel $0 \neq \tilde{q}(x, y) < K$ and $\tilde{R} = P^k - \tilde{Q}$. We have $\tilde{R}1 \neq 1$. There is no loss of generality in assuming that P^k is ergodic, because by theorem D Chapter V of [1] there exists a minimal set W and an integer d so that $X = W \cup PW \cup \cdots \cup P^{d-1}W$ and $P^d W = W$, and hence P^{jd+1} is ergodic for each j that $jd + 1 \geqq k$, but $Q_{jd+1} \geqq Q_k P^{jd+1-k} > 0$ and we can take instead of P^k, the ergodic operator P^{jd+1}. Let λ be a measure invariant under $\tilde{R}$, and hence $\lambda = \lambda\tilde{R} \leqq \lambda P^k \Rightarrow > \lambda = \lambda P^k$ but P^k is ergodic and therefore it has at most one invariant measure, hence $\lambda = \alpha\mu$ but μ is eigenvalent to m and $\tilde{R}1 \neq 1$ because $\tilde{Q}1 \neq 0$, hence $\mu\tilde{R}(1) = \mu(\tilde{R}1) < \mu(1)$, a contradiction. Hence there exists no measure invariant under $\tilde{R}$, and by Corollary 2 of Theorem E Chapter IV of [1], for each $\varepsilon > 0$. There exists a set A with $m(A^c) < \varepsilon$ so that:

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} \tilde{R}^k 1_A \right\|_\infty = 0. \tag{3.2}$$

Let ν be a positive pure charge invariant under P, then $\nu \geqq \nu\tilde{Q}$. Let $\{B_n\}$ be a decreasing sequence of sets so that $\bigcap_n B_n = \emptyset$, then

$$\nu\tilde{Q}(B_n) = \nu \int \tilde{q}(x, y) 1_{B_n}(y) m(dy) \leqq Km(B_n) \xrightarrow[n\to\infty]{} 0$$

hence $\nu\tilde{Q}$ is a measure and therefore $\nu\tilde{Q} = 0$. So we have $\nu = \nu P^k = \nu\tilde{R}$ and this inplies that if $f \in L_\infty(m)$ and $\operatorname{supp} f \subset A$ then $\nu(f) = 0$ by (3.2) and Theorem 1. Let $f \in L_\infty(m)$, orthogonal to μ and $\operatorname{supp} f \subset A$. Let ν an invariant charge, by [2] there is a decomposition $\nu = \nu_1 + \nu_2$ where ν_1 is a measure and ν_2 is a pure charge, now $\nu P = \nu_1 P + \nu_2 P$ and $\nu_1 P$ is a measure because P is defined on $L_1(m)$ hence $\nu_1 P \leqq \nu_1$ which implies $\nu_1 P = \nu_1$, and $\nu_2 P = \nu_2$, but P is ergodic, hence $\nu_1 = \alpha\mu$ thus $\nu_1(f) = 0$, on the other hand $\operatorname{supp} f \subset A$ implies $\nu_2(f) = 0$ hence $\nu(f) = 0$ and by Theorem 1 we have $\lim_{n\to\infty} \| 1/n \sum_{k=1}^{n} P^k f \|_\infty = 0$.

REFERENCES

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